NIKHEF/2003-014
IMAFF-FM-03/17
hep-th/0310295

On Orientifolds of c=1 Orbifolds

T.P.T. Dijkstra, B. Gato-Rivera[1], F. Riccioni[2] and A.N. Schellekens

NIKHEF
P.O. Box 41882 1009 DB Amsterdam
The Netherlands

Abstract

The aim of this paper is to study orientifolds of $c = 1$ conformal field theories. A systematic analysis of the allowed orientifold projections for $c = 1$ orbifold conformal field theories is given. We compare the Klein bottle amplitudes obtained at rational points with the orientifold projections that we claim to be consistent for any value of the orbifold radius. We show that the recently obtained Klein bottle amplitudes corresponding to exceptional modular invariants, describing bosonic string theories at fractional square radius, are also in agreement with those orientifold projections.

[1]Permanent address: Instituto de Matemáticas y Física Fundamental, CSIC, Serrano 123, Madrid 28006 Spain.

[2]New address: DAMTP, Wilberforce Road, Cambridge CB3 0WA, UK.

1 Introduction

The c=1 orbifold CFTs on closed, oriented Riemann surfaces have been studied extensively [1][2][3][4] because they provide simple but non-trivial examples of various features of conformal field theory. It has long been believed that all c=1 bosonic theories in the closed oriented case were known: they either belonged to the famous continuous moduli space of the circle and its Z_2 orbifold, or one of three discrete points discovered in [3]. The completeness proof [4] was based on certain assumptions, and more recently counterexamples have been conjectured to exist [5]. The open, unoriented case has received considerably less attention. Early results can be found in [6], reviewed recently in [7], and there are some remarks on such theories in the appendix of [8]. Our purpose here is a systematic study of the set of theories that is obtained if one allows open and unoriented surfaces. We will study this problem for continuous values of the radius R and at rational points, and match the results.

For the circle theories the solution to this problem is known. On oriented, closed surfaces the moduli space is a line parametrized by the radius R of the circle, with a T-duality identification of R and α'/R. On unoriented surfaces this line splits into two lines. In one of the two T-dual pictures the two lines are characterized by having O0 planes of either the same tension or opposite tension on opposite points of the circle, in the other the circle is covered by O1 planes. This matches precisely and unambiguously with the results from rational CFT, as we will see in slightly more detail below.

Studying this problem on the orbifold line is interesting for a variety of reasons: the set of modular invariant partition functions (MIPFs) is richer, there is an interesting one-to-one map between the rational c=1 orbifolds at radius $R^2 = \alpha' N$ and the D_N WZW-models at level 2 [9] and the orbifolds have a twist field degeneracy [2] which is resolved only in the rational points. Finally we are interested in understanding the puzzling results of [10], showing that in certain rational points the orbifold CFT does not admit a canonical Klein bottle projection (with all coefficients equal to 1). There are non-canonical solutions, but it is not immediately clear how they would fit into a continuum description.

Two types of approaches to boundary RCFT are used in this paper. In [11] a general formula was given for all reflection coefficients and crosscap coefficients for all simple current modular invariants [12] and a large class of orientifold projections. This formula is consistent (at least in the sense of yielding integral coefficients, see [11] and [13]; further-

more consistency for all resulting oriented amplitudes was demonstrated recently in [14]), but there is no proof that it is complete. In [15] a set of polynomial equations was written down which can be solved for annulus, Moebius and Klein bottle coefficients. This method is complete, but it is not known if all solutions are physical. Our goal is to do a systematic analysis of all known orbifold RCFT's and compare them with the continuum, to check if the method of [11] misses anything, and if the method of [15] produces anything manifestly unphysical. The orbifolds are an ideal laboratory for this because the continuum can be obtained explicitly, and RCFT's are known in all rational points. The circle has all these features as well, but lacks some non-trivial structure related to the fixed points, as well as exceptional modular invariants.

A prerequisite for this work is a complete description of orientifolds in the continuous case. Although there is a large body of work of orientifolds of tori and orbifolds, as far as we know the complete and explicit answer for the $c = 1$ orbifolds is not available yet, although a partial result can be found in [7]. We find a total of four different allowed Klein bottle amplitudes and corresponding orientifold projections.

The structure of the paper is the following. In Section 2 we review the results concerning the bosonic string on a circle and the orientifolds thereof. In Section 3 we consider the orbifold case. First, in subsection 3.1, we study orientifolds for rational radius. A parametrization of the allowed Klein bottles in terms of RCFT characters leads to twelve different choices. Then we enumerate the modular invariant partition functions (MIPFs), by systematically combining all known types and checking closure under multiplication. For each MIPF we perform a systematic construction of U-NIMreps [16] (U-NIMreps are sets of Annulus, Möbius and Klein bottle amplitudes that satisfy the aforementioned polynomial equations; they are NIMreps [17][18][19] extended with data concerning unoriented surfaces.) This method is limited in practice to a finite number of primaries, but we can extend it far enough to uncover the complete picture, and furthermore we supplement it with the formula of [11], which has no such limitation. We conclude that for integer values of R^2/α' six different cases can be distinguished, corresponding to four distinct continuum Klein bottle amplitudes. As a result of these computations we obtain a set of boundary and crosscap coefficients which we use in subsection 3.1.4 to study the localization of branes and O-planes, as well as the Chan-Paton groups in the various cases. In subsection 3.2 we study the case of orientifolds at continuous radius from a geometric point of view, and confirm that the four Klein bottle amplitudes are indeed the only possible ones. In subsec-

tion 3.3 we consider the case of certain non-integer rational radii (exceptional MIPFs) and show that, despite their unexpected features, the results of [10] are precisely in agreement with the four orientifold projections. In Section 4 we summarize our conclusions. In the appendix we discuss a variety of orbifold maps needed to obtain distinct orbifold theories that exist in the rational case.

2 Summary of circle results

In this section we want to review know results about orientifolds of the bosonic string on a circle, in order to set up the discussion for the orbifold case and to introduce some notations. Orientifolds of circle compactifications for irrational values of the radius appeared for the first time in [20], while additional references are [21][22][23][24][25][26] (see [7] for a review). The partition function for a bosonic closed oriented string compactified on a circle is

$$Z(R) = \frac{1}{\eta\bar{\eta}} \sum_{m,n} q^{\frac{\alpha'}{4}(m/R+nR/\alpha')^2} \bar{q}^{\frac{\alpha'}{4}(m/R-nR/\alpha')^2}, \tag{2.1}$$

where we denote with n the winding and m/R the Kaluza-Klein (KK) momentum along the circle. A well known feature of this partition function is the fact that it is invariant under the exchange $R \leftrightarrow \alpha'/R$. From the point of view of the extended CFT that describes the above theory, the partition function $Z(R)$ hides chiral information. For each value of R there are actually two theories with the same partition function, which are each others T-dual, one with momentum states $|\sqrt{\frac{\alpha'}{2}}(m/R + nR/\alpha'), \sqrt{\frac{\alpha'}{2}}(m/R - nR/\alpha')\rangle$, and one with states $|\sqrt{\frac{\alpha'}{2}}(m/R + nR/\alpha'), \sqrt{\frac{\alpha'}{2}}(-m/R + nR/\alpha')\rangle$. The first of these is obtained in a genuine compactification on a circle of radius R.

The allowed orientifold projections must respect the operator product expansion of the CFT (we will only consider orientifold projections of order 2). This implies in particular that the operator $\partial X\bar{\partial}X$ must be transformed into itself, and that the vertex operators corresponding to the momentum and winding states must transform with a factor $(\epsilon_1)^m(\epsilon_2)^n$, with ϵ_1 and ϵ_2 equal to ± 1. Hence one can associate four consistent Klein bottles with this

partition function, namely[3]

$$\frac{1}{\eta(2i\tau_2)}\sum_m q^{\frac{\alpha'}{2}(m/R)^2} \equiv 2K_{+00}(R) \tag{2.2a}$$

$$\frac{1}{\eta(2i\tau_2)}\sum_m (-1)^m q^{\frac{\alpha'}{2}(m/R)^2} \equiv 2K_{-00}(R) \tag{2.2b}$$

$$\frac{1}{\eta(2i\tau_2)}\sum_n q^{\frac{1}{2\alpha'}(nR)^2} \equiv 2K_{0+0}(R) \tag{2.2c}$$

$$\frac{1}{\eta(2i\tau_2)}\sum_n (-1)^n q^{\frac{1}{2\alpha'}(nR)^2} \equiv 2K_{0-0}(R)\ . \tag{2.2d}$$

The functions K will be introduced later. The Klein bottle amplitude is subject to the constraint that $\frac{1}{2}(Z(R)+K(R))$ expands into non-negative integers. The first two Klein bottles can be combined with the diagonal theory, the other two with its T-dual. The positions of the orientifold planes can be derived from the transverse channel Klein bottle amplitudes by dimensional analysis. In the first two cases the transverse amplitude describes a closed string propagating between two O1 planes. There are two different O1 planes. In the first case, the resonance term permits only to even winding states to propagate in the transverse channel, while in the second case only odd winding states contribute, and thus the configuration has vanishing tension, since the graviton does not propagate in the transverse channel. We call these two configurations $\mathrm{O1}_+ \oplus \mathrm{O1}_+$ and $\mathrm{O1}_+ \oplus \mathrm{O1}_-$ respectively. In the first case, the orientifold projection maps X to itself, while in the second case it maps X to $X+\pi R$. Both these maps square to the identity because of the periodicity of the circle, and have no fixed points. The other two cases are the T duals of the former. After T duality, the orientifold projection acts as a Z_2 orbifold on the circle coordinate, so that the model lives on a segment, with O0 planes at the endpoints. In the third case, the orientifold projection maps X to $-X$, while in the fourth case it maps X to $-X+\pi R$. Both these projections have fixed points, where the O0-planes are located. More precisely, the third case corresponds to the configuration $\mathrm{O0}_+ \oplus \mathrm{O0}_+$, in which the two O-planes have the same tension, while the fourth case corresponds to $\mathrm{O0}_+ \oplus \mathrm{O0}_-$, in which the O-planes have opposite tension.

[3]We assume here that the relation between the orientifold projection and the Klein bottle is straightforward. This implies in particular that the Klein bottle coefficients are preserved under fusion, *i.e* that the "Klein bottle constraint" is satisfied. There are examples where this constraint is not satisfied (see *e.g* [27]) and which require further thought, but this problem does not occur for any of the c=1 U-NIMreps.

The corresponding rational CFTs are obtained by setting $R^2 = \alpha' N$. All primaries of this CFT are simple currents, forming a Z_{2N} discrete group. We denote the generator of Z_{2N} as J. All MIPFs are simple current invariants, and they are in one-to-one correspondence with the subgroups of Z_{2N} generated by even powers of J, which in their turn are in one-to-one correspondence with the divisors of N. If m is a divisor of N, the MIPF belonging to the subgroup generated by J^{2m} corresponds to a circle with radius $R^2 = \alpha' N/m^2$. We will refer to this partition functions as $Z_{\rm circle}(m, N)$, with the convention that $Z_{\rm circle}(N, N)$ is the charge conjugation invariant and $Z_{\rm circle}(1, N)$ the diagonal one. For every rational radius $R^2 = \alpha' p/q$ (with p and q relative prime) there is an infinite number of rational CFTs describing it, namely $Z(pr, pqr^2)$, for any $r \in \mathbf{Z}$. For $r > 1$ the corresponding partition functions involve extensions of the chiral algebra.

For all these MIPFs the allowed crosscap and boundary coefficients follow from the general formula presented in [11] (summarizing and extending earlier work in [28]), which in the special case $Z_{N,N}$ reduces to the well-known boundary state of Cardy [29] combined with the crosscap state due to the Rome group [30].

By Fourier analyzing the closed string scattering amplitudes from the boundary and crosscap states (a procedure that was pioneered in [31], and applied in [32] to boundary states of WZW models and in [33] to crosscap states (see also [34][35])) one can localize the D-branes and O-planes on the circle. To do this one multiplies the boundary and crosscap amplitude with a factor $e^{ikx/R}$ and sums over all values of k in the primary range $-N \leq k < N$. The resulting function of x has peaks that get more pronounced with increasing N, and are interpreted as the brane and plane positions. ¿From the continuous point of view, we expect D1 branes for the diagonal invariant (corresponding to a genuine circle compactification, with the space-filling brane wrapped around the circle), which turn into D0 branes for the charge conjugation invariant, the T-dual of the former.

In the rational CFT one finds the following. The MIPF $Z(m, N)$ admits $2m$ boundaries, each of which is localized at $n = N/m$ distinct points on the circle. To make sense of this we introduce the dual radius $\hat{R} = \alpha'/R$, which is the relevant quantity because D0 branes live on the dual circle. The n-fold multiplicity is an indication of the fact that the original circle of dual radius $\hat{R}$ is to be interpreted as an n-fold cover of a circle of radius $\hat{R}/n$. We may label the boundaries by integers $a = 0, \ldots, m-1$, such that they are localized at points $\left(\frac{a+2m\ell}{2N}\right) 2\pi\hat{R}$, $\ell = 0, \ldots, n-1$. For $m = N, n = 1$ (the charge conjugation invariant) this yields $2N$ D0 branes equally distributed over the circle; for $m = 1, n = N$ (the diagonal

invariant) this gives two branes localized simultaneously on N equally distributed points, one brane on the odd points and one on the even points. This is the RCFT realization of a D1 brane.

In the continuum the D0 can be localized anywhere on the circle, whereas in the rational CFT description their positions are quantized. One can approach the continuum results either by using deformations of the boundary CFT [36], or by allowing boundaries that break some of the extended symmetries that characterize the rational CFT. This can be done by obtaining the circle at some dual radius $\hat{R}$ from a circle at dual radius $\hat{R}r$ by extending the CFT of the latter. The RCFT notion of "completeness of boundaries" [37], when applied to MIPFs of extension type, automatically implies the presence of boundaries that break the extended symmetries. Indeed, if we consider $Z(pr, pqr^2)$ we find $2pr$ distinct boundaries, each localized in qr points on the circle of radius $\hat{R}r$. This circle is an r-fold cover of the circle of radius $\hat{R}$, so that on the latter circle we now have $2pr$ distinct branes each localized in q points. Of these, $2p$ coincide with the ones found for $r = 1$, and the remaining ones occupy intermediate positions. In the limit $r \to \infty$, $\hat{R}$ fixed we approach the continuum result.

Similar results hold for crosscaps. The formalism of [11] allows two ways of modifying the crosscap coefficient for a given MIPF. The formula for the crosscap coefficient is, up to normalization

$$\Gamma_i \propto \sum_{L \in G} \eta(K, L) P_{KL,i} \ , \tag{2.3}$$

where i is the Ishibashi label, P the P-matrix ($P = \sqrt{T}ST^2S\sqrt{T}$). Here K is a simple current (subject to a condition given below), $\eta(K, L)$ a set of signs satisfying the constraint

$$\eta(k, L) = e^{\pi i(h_K - h_{KL})} \beta_K(L) \quad , \tag{2.4}$$

where $\beta_K(L)$ is a set of phases solving the relation

$$\beta_K(LJ) = \beta_K(L)\beta_K(J)e^{-2\pi i X(L,J)}, \tag{2.5}$$

where X is the rational bihomorphism that specifies the MIPF, as defined in [12]. This relation does not fix the phases completely: for every independent even cyclic factor of the simple current group G, there is a free sign. These free signs are called the "crosscap signs". The current K (for historical reasons called the "Klein bottle current") must be local w.r.t. the currents of order two in G, and currents that differ by elements of G or

by squares of simple currents yield equivalent crosscaps. These Klein bottle currents form, together with the crosscap signs, the set of allowed crosscap modifications.

In the present case, it is not hard to see that for each choice of G there are just two solutions. If G has even order, there is a single crosscap sign choice, but there are no Klein bottle currents local w.r.t. G. If G has odd order n there is no crosscap sign choice, but then the current J^n is local w.r.t. G and is a non-trivial Klein bottle current. These two choices correspond precisely to the two orientifold choices in the continuous case. In all cases one of the orientifold choices leads to Klein bottle coefficients that are equal to $+1$ for all fields appearing diagonally.

The crosscap positions can be worked out in the same way as for D-branes. For the MIPF $Z(m, N)$ we find that a crosscap state occupies $2n$ positions, twice as many as a boundary state. These positions are n-fold identified. In the simplest case, $m = N, n = 1$ there are two positions, diametrically opposite on the circle. In this case, the crosscaps are characterized by the choice of Klein bottle current $K = J^k$. Each k corresponds to two O0 planes localized at $r = \frac{k}{4N} 2\pi R$ and $r = (\frac{k+2N}{4N}) 2\pi R$. If $k + N$ is even these two O0 planes have the same tension, if $k+N$ is odd they have opposite tension. If k is even, the O-plane locations coincide with a brane position; if k is odd the O-planes lie between two brane positions. Configurations where k differs by an even integer can be obtained from each other by rotating the circle, in agreement with the fact that the corresponding Klein bottle currents are equivalent. The T-dual configuration corresponds to the diagonal invariant, obtained by using the simple current J^2. This MIPF admits just two Ishibashi states, and hence only two non-vanishing crosscap coefficients. This is not sufficient to contain any information about localization, in agreement with the fact that we expect the O-planes to be O1 planes wrapping the circle (and analogously for boundaries).

3 Orbifolds

We are considering the $c = 1$ case, that is the real line modded out by the group G of reflections and translations, resulting in the segment $\mathbb{R}/G = S^1/\mathbb{Z}_2$. Following [38] we will denote the action of elements of this group on the string coordinate as

$$(\theta, n) \in G \ , \quad (\theta, n)X = \theta X + 2\pi n R \ , \qquad n \in \mathbb{Z} \ , \ \theta = \pm \ . \tag{3.1}$$

Strings on an orbifold are closed if they are periodically identified *up to* an element of this group:

$$X(\sigma + 2\pi) = (\theta, n)X(\sigma) \ . \tag{3.2}$$

X is then twisted by the element (θ, n); this defines various sectors with different periodicity conditions on X. Not all elements of G give rise to a different sector, a sector twisted by g being the same as the one twisted by $h\,g\,h^{-1}$. Thus we get a sector for each conjugacy class. One has the following conjugacy classes: $(+, \pm|m|)$, $(-, \text{even})$ and $(-, \text{odd})$. The first gives the circle periodicity conditions with winding number m, where now the winding direction is no longer significant. The last two cases give the twisted sectors. Note that in these sectors the notion of winding is limited to being 'even' or 'odd'.

In the untwisted sector, X has the same mode expansion as in the circle, with the difference that now only the states that are invariant under the map $X \to -X$ are present. Denoting with r the operator that performs this map on the Hilbert space, the resulting spectrum is obtained by applying the projector $\frac{1+r}{2}$ on the circle states.

In the twisted sectors the mode expansion for X is

$$X = x_o + \sum_n \frac{1}{n - \frac{1}{2}} \left(a_{n-\frac{1}{2}} z^{-n+\frac{1}{2}} + \bar{a}_{n-\frac{1}{2}} \bar{z}^{-n+\frac{1}{2}} \right) \ , \tag{3.3}$$

where $x_o = 0$ for the $(-, \text{even})$ sector, $x_o = \pi R$ for the $(-, \text{odd})$ sector. The twisted sectors correspond thus to states localized at the fixed points of the orbifold. Also the states created by the modes in (3.3) must be projected by the operator $\frac{1+r}{2}$.

The resulting partition function is

$$Z_{\text{orb}}(R) = \frac{1}{2} Z_{\text{circle}}(R) + \left| \frac{\eta}{\theta_2} \right| + \left| \frac{\eta}{\theta_3} \right| + \left| \frac{\eta}{\theta_4} \right| \ . \tag{3.4}$$

The first two are contributions of the untwisted sectors, while the last two are contributions of the twisted sectors.

3.1 Orientifolds for rational radius

3.1.1 Parametrizations of the Klein bottle

In the CFT description, the allowed orientifold projections are limited by the requirement of preservation of the OPE. Of most interest are the projection signs of states appearing diagonally, since those signs affect the Klein bottle. Again we find that $\partial X \bar{\partial} X$ must

transform into itself. This implies that $\frac{\eta}{\theta_2}$ cannot appear in the Klein bottle expression, since this function represents the *difference* of contributions of the identity operator and the operator $\partial X\bar{\partial}X$. The OPE of $\partial X\bar{\partial}X$ with a lowest weight twist field $\sigma(z,\bar{z})$ yields the excited twist field $\tau(z,\bar{z})$. Since $\partial X\bar{\partial}X$ has projection sign +, the twist fields σ and τ must be projected in the same way. This implies the absence of $\frac{\eta}{\theta_3}$, which corresponds to the difference of the two twist field labels.

An important issue is twist field degeneracy. The $c = 1$ orbifold has two twist fields (stemming from the fact we have two different twisted sectors), denoted σ_1 and σ_2 (with weight $(\frac{1}{16},\frac{1}{16})$) and two excited twist fields, τ_1 and τ_2, with weight $(\frac{9}{16},\frac{9}{16})$. The labels 1 and 2 are not distinguished by the Virasoro algebra. On any point on the orbifold line the Virasoro algebra is extended by operators that are even polynomial in ∂X and its derivatives, the first one at weight 4 [2]. But these operators do not distinguish the labels either, since ∂X itself does not. Only in the rational points there are operators that distinguish the twist fields, namely the operators $\cos(\sqrt{\frac{2}{\alpha'}}mRX)$ that extend the chiral algebra to make the CFT rational. Hence we should regard these as states with multiplicity 2. The allowed Klein bottle coefficients in this sector are the $2, 0$ or -2. The value 0 is allowed if the twist fields appear off-diagonally in the partition function, or if they appear diagonally, but have opposite Klein bottle projections. Based on this information we arrive at the following twelve choices for the Klein bottle

$$K_{\epsilon_1\epsilon_2\epsilon_3} = \frac{1}{2}\frac{1}{\eta(2i\tau_2)}\sum_{k\in\mathbf{Z}}(\epsilon_1)^k q^{\frac{\alpha'}{2}\left(\frac{k}{R}\right)^2} + \frac{1}{2}\frac{1}{\eta(2i\tau_2)}\sum_{m\in\mathbf{Z}}(\epsilon_2)^m q^{\frac{1}{2\alpha'}(mR)^2} + 2\epsilon_3\sqrt{\frac{\eta}{\theta_4}}\,, \qquad (3.5)$$

with $\epsilon_1 = \pm 1; \epsilon_2 = \pm 1$ and $\epsilon_3 = 0, \pm 1$. The same parametrization can be used for the circle theory, provided one allows the value 0 for ϵ_1 and ϵ_2.

As was the case for the circle, the allowable Klein bottles depend on the interpretation of the partition function. But in contrast to the circle theory, this interpretation is discontinuous in R. This is due to the fact that we can distinguish the twist fields only for rational R. In addition, for rational R two orbifold fields appear that do not exist for irrational values of R, namely the fields we denote as ϕ_1 and ϕ_2 and that have conformal weight $\frac{1}{4}N$.

3.1.2 Enumeration of modular invariants

We will now study the orbifold at rational points in order to reduce the set of orientifold projections. In the rational points and for a sufficiently small set of primaries we have an

additional tool at our disposal, namely the systematic search for NIMreps and U-NIMreps.

The orbifold of the circle of radius $R^2 = \alpha' N$ (or its T-dual) is the well-known orbifold rational CFT with $N + 7$ primaries. It has four simple currents, 1, ϕ_1, ϕ_2 and the spin-1 current ∂X, forming a discrete group $\mathbf{Z}_4$ (for N odd) or $\mathbf{Z}_2 \times \mathbf{Z}_2$ (for N even). The remaining fields will be denoted φ^k, $k = 1, \ldots, N_1$, $\sigma_i, \tau_i (i = 1, 2)$, following [2]. A lot is known about the MIPFs of these CFTs, but the result are scattered throughout the literature, and for that reason we will give here an enumeration of what is known.

In [2] it was observed that the theory at radius $R^2 = \alpha' p/q$ or $R^2 = \alpha' q/p$ has the same chiral algebra as the one at $R^2 = \alpha' pq$. Hence it is described by a non-trivial MIPF of the theory at $R^2 = \alpha' pq$. This MIPF is of exceptional type, except when q and/or p is equal to 2, in which case the invariant is of simple current type. Just as in the circle case, one can generalize this by allowing p and q to have common factors. In this way one can obtain an infinite number of rational CFT realizations at any rational point on the orbifold line. Any of these MIPFs can be extended by the simple current ∂X to re-obtain the circle partition function.

But there are still more rational partition functions for every rational point, a fact that can most easily be appreciated by using the fact that the modular group representation of the orbifold CFT for $R^2 = \alpha' N$ is in one-to-one correspondence with the D_N WZW model at level 2 [9]. In particular there is a one-to-one relation for partition functions, NIMreps and U-NIMreps. While the aforementioned MIPFs describing orbifolds at non-integer radii do not seem to have a *raison d'etre* for the WZW-models, they do exist for these models as well. The ones of automorphism type were discovered using Galois symmetry in [39] (subsequently the WZW automorphisms were fully classified in [40]); the extensions were described in [9] using the aforementioned correspondence. On the other hand, the D_N WZW models (at any level) have MIPFs related to Dynkin diagram automorphisms that imply the existence of related invariants for the orbifolds. These are first of all the conjugation invariants, which have an off-diagonal pairing of ϕ_1, ϕ_2, σ_1, σ_2 and τ_1, τ_2. For odd N this is the charge conjugation invariant, which can also be described as a simple current invariant generated by ϕ_1 (or, equivalently, ϕ_2). For even N this is an exceptional invariant (for even N, the simple current invariant generated by ϕ_1 gives an orbifold at reduced radius $N/4$). If the MIPF involves an extension by ϕ_1 or ϕ_2 (which happens if p and q are both even), there are still more possibilities, because one can then conjugate the left and right chiral algebras independently. As a result one obtains two symmetric and

two asymmetric (heterotic) MIPFs. Finally, for $N = 4$ there are even more MIPFs related to triality of D_4; there are 16 MIPFs in total.

For given $N \neq 4$ the number of known modular invariants, obtained by combining all the above, can be described as follows. Let p be a divisor of N in the range $1 \leq p \leq \sqrt{N}$, and define $q = N/p$. The number of known invariants is equal to $\sum_p M(p)$ where the sum is over all divisors p in this range, and $M(p) = 3$ if either p or q is odd, $M(p) = 5$ otherwise. The multiplicity three corresponds to the diagonal invariant, the conjugation invariant and the circle extension, which we denote respectively as $Z_D(p, N)$, $Z_C(p, N)$ and $Z_X(p, N)$. The multiplicity five corresponds to the four cases described above plus the circle extension, denoted respectively as $Z_{11}(p, N), Z_{22}(p, N), Z_{12}(p, N), Z_{21}(p, N)$ and $Z_X(p, N)$ (the circle extension includes both ϕ_1 and ϕ_2).

Since the standard orbifold map $X \to -X$ yields just one orbifold theory for each T-dual pair of circle theories, one has to consider more general orbifold maps to get the various types of orbifold partition functions. These maps are discussed in the appendix.

3.1.3 U-NIMreps

Consider first $Z_C(1, N)$ and $Z_D(1, N)$. In all but one case these MIPFs are C-diagonal or of simple current type, and a set of Klein bottle coefficients can be obtained from various previous papers. To deal with the remaining exceptional MIPF ($Z_C(1, N)$, N even) , and as a check on all the others, we solved the U-NIMrep polynomial equations (for $N \leq 16$) to get the complete answer. This results in the following six cases:

1. The diagonal invariant, with standard Klein bottle ($K_i = 1$ for all i). For N even, this is the standard Cardy-Rome case. For N odd, it is a simple current automorphism, as treated in [11], with suitable choice of the crosscap sign. The resulting Klein bottle amplitude is K_{+++}. This result was first obtained in [6] (see also [7]).

2. The diagonal invariant, with Klein bottle coefficients -1 in the twisted sector. For N even, this is the same as the previous case but with Klein bottle current 2. For N odd, it is the same as the previous case, but with the opposite crosscap sign. The resulting Klein bottle is K_{++-}.

3. The diagonal invariant, with Klein bottle currents ϕ_1 or ϕ_2. This case exists only for N even. For N odd, the diagonal invariant is generated as a simple current automorphism of current ϕ_1, and the only allowed sign changes are the crosscap

signs, which we already saw above. For even N one finds that all odd charged fields φ^k get a negative Klein bottle, *i.e.* $K_{\varphi^k} = (-1)^k$. Furthermore either σ_1 and τ_1 or σ_2 and τ_2 change sign, so that the total twisted sector contribution cancels. The result is K_{-+0} (for N even only).

4. The charge conjugation invariant with standard Klein bottle (*i.e.* all $K_i = +1$ if i appears diagonally). The charge conjugation invariant only differs from the diagonal one for N odd. The effect is that ϕ_1, ϕ_2 appear off-diagonally, and the same for the twisted sector. The latter implies $\epsilon_3 = 0$. The absence of ϕ_1, ϕ_2 in the Klein bottle amplitude implies $\epsilon_2 = -1$, so that the contribution of ϕ_i cancels between the first two terms. Hence we get K_{+-0} (for N odd only).

5. The charge conjugation invariant with non-standard Klein bottle (case 4 with simple current Klein bottle current ϕ_1, which is equivalent to ϕ_2). This gives a sign flip for all odd charge fields, implying $\epsilon_1 = -1$ This can be taken into account by inserting a $(-1)^k$ into the first sum. Since N is odd, the ϕ_i contribution cancels between the first two terms if and only if $\epsilon_2 = +1$. Hence we get K_{-+0} (for N odd only).

6. The conjugation invariant for even N. This is an exceptional invariant that pairs ϕ_1 with ϕ_2, σ_1 with σ_2 and τ_1 with τ_2. Here [11] does not apply, but by solving the NIMrep conditions explicitly we find only one NIMrep with one U-NIMrep. The Klein bottle has all allowed coefficients equal to 1. This yields K_{+-0} (for N even only).

We summarize these results in the following table. In the first column 'D' denotes the diagonal invariant, 'C' the charge conjugation invariant, and 'T' the twist field conjugation invariant, in which ϕ_i, σ_i and τ_i are off-diagonal. In the fifth column we indicate the Chan-Paton group for the dominant branes (*i.e.* the ones that are most numerous for large N). This will be explained in the next subsection. The last column refers to the six cases listed above.

Invariant	N	Boundary/Crosscap formula	Klein bottle	CP-group	case
D	odd	[11]	$+++$	SO	1
			$++-$	SO	2
C=T	odd	Cardy/Rome	$+-0$	SO	4
			$-+0$	U	5
D=C	even	Cardy/Rome	$+++$	SO	1
			$++-$	SO	2
			$-+0$	U	3
T	even	exceptional	$+-0$	SO	6

Note that all allowed continuous Klein bottle amplitudes make their appearance for both odd and even N, but in rather different ways. Note also that for even N the diagonal invariant (D) allows four different orientifold projections (the case K_{-+0} actually consists of two subcases with opposite signs for all Klein bottle coefficients in the twisted sector), whereas the twist conjugation invariant (T) allows only one. This is strange because we expect these theories to be dual to each other (in the sense of the existence of a one-to-one map between their operators, respecting all correlators). This duality is of course not the T-duality of the circle (which was modded out in the orbifold). We do not know if such a duality has been proved in the literature, but it certainly seems to hold in the simplest case, $N = 2$, the tensor product of two Ising models. Note that T-dual circles admit the same number (namely two) of O-plane/D-brane configurations, and the only aspect that differs is the number of allowed D-brane positions. In the orbifold case two probably dual theories have a different number of orientifold projections, corresponding to physically different configurations, with different CP groups. Although this is counterintuitive, on the other hand it does not seem to contradict the duality in an obvious way. Note that also the number of boundary conditions differs for the two mutually dual cases, but this merely corresponds to a different choice of rationally allowed positions for the same D-branes. Note furthermore that for T-dual rational circle theories the number of orientifold choices is the same.

After this enumeration (which is exhaustive for small N) only four of the twelve potential Klein bottles are realized. Most absences can be explained by a combination of the following facts

- The twist fields σ_i, τ_i and the fields ϕ_i must be simultaneously (off)-diagonal in any MIPF. This follows from modular invariance.

- The Klein bottle coefficients of ϕ_1 and ϕ_2 must be identical, because these fields are either each others conjugates, or they fuse to $\partial X \bar{\partial} X$, which must have projection sign 1. We call these coefficients K_ϕ.
- K_ϕ can be expressed as $\frac{1}{2}(\epsilon_2 + \epsilon_1^N)$.
- The fusion of σ_1 and σ_2 produces fields φ^k, with $k + N$ odd.
- The fusion of σ_1 and τ_1 produces fields φ^k, with $k + N$ even.

The last two points are relevant only if the twist fields appear diagonally. These points imply the following. For conjugation invariants we must have $\epsilon_3 = 0$ and $\epsilon_2 = -\epsilon_1^N$. For diagonal invariants we must have $\epsilon_2 = \epsilon_1^N$. Furthermore from point 5 we find that $\epsilon_1 = 1$ for N odd. Hence $\epsilon_2 = 1$ for all diagonal invariants. If $\epsilon_3 = 0$ the projections of σ_1 and σ_2 are opposite. This is impossible for N odd, and implies $\epsilon_1 = -1$ for N even. On the other hand, if $\epsilon_3 \neq 0$, point 4 implies $\epsilon_1 = \epsilon_2 = 1$ for N even.

This only leaves one case that was not found, and is also not yet ruled out, namely $\epsilon_1 = \epsilon_2 = -1$ for conjugation invariants with even N. This case can be ruled out by computing the transverse channel amplitude. It turns out that ϕ_1 and ϕ_2 propagate in the transverse channel, although they are not Ishibashi states. Hence this case must be rejected.

As an additional check on the result one can now solve the U-NIMrep equations for all other accessible MIPFs as well. The results are in complete agreement with the foregoing: $Z_D(p, N)$ has 4 U-NIMreps for N even, 2 for N odd; $Z_C(p, N)$ has one U-NIMrep for N even, and 2 for N odd; $Z_{ii}(p, N)$ always has 4 U-NIMreps, whereas $Z_{ij}(p, n), i \neq j$, has none. Finally $Z_X(p, N)$ always has 4 NIMreps, except when $N = p^2$, in which case it has two. The four U-NIMreps of Z_X correspond precisely to two distinct Klein bottle choices for the circle and its dual. For $N = p^2$ one ends up in the self-dual point, which explains why one gets only half the number of solutions. In all cases these distinct U-NIMreps correspond to choosing different boundary conjugations for a single NIMrep, although $Z_C(3, 9)$ and $Z_D(3, 9)$ have respectively one and two additional NIMreps that do not admit any U-NIMreps, and are presumably spurious. Finally, $Z_{ij}(p, n), i \neq j$ was found to have a single NIMrep which does not admit a U-NIMrep, in agreement with the fact that these MIPFs are asymmetric.

The U-NIMreps for $Z_C(p, pq)$ and $Z_D(p, pq)$, with p and q prime have been given explicitly in [10], and U-NIMreps for simple current MIPFs are described in [11]. In all

other cases these conclusions, as well as the completeness of the entire picture, rely on extrapolation to arbitrary N.

3.1.4 Localization and Chan-Paton groups

In the rational CFT description one can attempt to get information about the boundary and crosscap states by analyzing the Fourier transformation of their coupling to closed string states. This amounts to probing the brane/plane positions by scattering of gravitons [31] (or, equivalently, dilatons or tachyons). This method has a clear physical interpretation in flat space, but becomes less intuitive when applied to compact spaces, although sensible results are obtained for the circle (as explained above) and WZW-models [32] [33]. Apart from the proper physical interpretation, a second caveat is that this method requires a precise knowledge of the boundary and crosscap coefficients. In many cases the latter are obtained by imposing integrality conditions on annulus, Möbius and Klein bottle amplitudes. These amplitudes are not sensitive to sign changes in the coupling to Ishibashi states provided one makes the same sign change in the boundary and the crosscap coefficients. Such sign changes do not affect tadpole cancellation either. In principle the true sign can be determined by solving the sewing constraints, but that has been done only in very few cases. However, in the Cardy case the results of [41] imply that the signs are correct. This should then also apply to all possible choices of orientifold projections, since this is expected to add O-planes in different positions while keeping the branes fixed.

Keeping these caveats in mind, we can compute the positions as follows. The coupling to the fields φ^k provides a natural set of Fourier components for the couplings. Inspired by the circle results we define a shape function

$$F(x) = \sum_{k=1}^{N-1} (e^{ikx/R} + e^{-ikx/R}) C(k) \ , \tag{3.6}$$

where $C(k)$ is a boundary or crosscap coefficient[4] for the coupling to φ^k. This function is periodic with period $2\pi R$ and symmetric in $x \to -x$ as well as $\pi R + x \to \pi R - x$, and therefore it is natural to identify the line segment $[0, \pi R]$ with the orbifold. Note that in the diagonal and conjugation modular invariants of the orbifolds with $R^2 = \alpha' N$ all Ishibashi labels k occur, so that there are no other points with reflection symmetry: the

[4]We use here the coefficients specified in [11], but without the denominator factors $\sqrt{S_{Ki}}$. It is more natural to absorb these factors in the Ishibashi metric for the unoriented annulus, so that the boundary coefficients themselves are independent of the choice of orientifold. See also [32], [33] and [10].

identification of the two orbifold points is unambiguous. By defining $C(-k) = C(k)$ the second term can be used to extend the sum to negative k. The coefficients $C(0) - C(\partial X)$ and $C(\phi_1) + C(\phi_2)$ turn out to have precisely the right value to complete the sum to the range $-N \leq k < N$. The resulting function $F(x)$ typically has one or two positive or negative peaks along the orbifold line, which approach δ-functions for large N. We interpret the extrema as O0 plane positions, and the sign as an O0 charge. For the six cases in the table, the coefficients $C(k)$ for the crosscaps either vanish for all even k, or for all odd k. The non-vanishing values are all equal to $1/\sqrt{N}$, up to a sign. If this sign is positive, this leads respectively to opposite charge or same-charge planes at the two endpoints of the orbifold line, $x = 0$ and $x = \pi R$. The other possibility we encounter is a sign $(-1)^{k/2}$ for even k. This shifts the plane positions by $\frac{1}{2}\pi R$, so that they are on top of each other.

Because the orbifold incorporates the circle T-duality, which interchanges D0(O0) with D1(O1) branes (planes) we expect boundary and crosscap states to describe a combination of branes and planes of dimension 0 and 1. While the D0/O0 positions and charges can be extracted very easily from $F(x)$, this is not the best way to determine the D1/O1 charges. The information is in fact hidden in the linear combinations $C(0) + C(\partial X)$ and $C(\phi_1) - C(\phi_2)$ which are not used in the computation of $F(x)$. A Fourier transform of these two quantities yields identical values on all allowed brane positions in the first case, and alternating values on even and odd positions in the second case. Furthermore in all cases either $C(0)+C(\partial X)$ or $C(\phi_1)-C(\phi_2)$ is zero. Remembering how D1 branes emerged for the rational circle, we are led to the conclusion that a non-vanishing $C(0) + C(\partial X)$ implies the presence of two equal-charge D1/O1 branes/planes, whereas a non-vanishing value of $C(\phi_1) - C(\phi_2)$ implies two opposite-charge D1/O1 branes/planes.

The "charge" referred to above always refers back to the corresponding quantity for the circle, namely the dilaton coupling strength. This allows us to interpret any orbifold brane/plane configuration in terms of a collection of circle configurations of different dimension and charges. Not surprisingly, this interpretation breaks down for branes labelled by twist fields, that have no circle analog. Furthermore the values $C(\sigma_i)$ and $C(\tau_i)$ (which vanish for crosscaps and most boundaries) are also not used, and we do not have a geometric interpretation for these values.

Some more information about brane and plane positions can be gathered from boundary conjugation, which geometrically corresponds to a reflection of a brane through an

O-plane. This property affects the Chan-Paton groups of the brane, which is orthogonal or symplectic for self-conjugate (real) branes and unitary for pairs of conjugate branes. Boundary conjugation and the allowed CP-groups are not affected by the aforementioned sign ambiguities, but the distinction between symplectic and orthogonal for real boundaries does depend on the overall sign of all crosscap coefficients relative to all boundary coefficients. This sign determines the O-plane tension, and whenever we specify a CP group below we have fixed the tension to a negative value, so that the dilaton tadpole can be cancelled (in principle) between D-branes and the O-plane. Orbifold O-planes (unlike circle O-planes) always have non-zero tension.

For the cases discussed listed in the table we find the following positions:

- N even, Diagonal invariant: This is the Cardy case, so boundary labels correspond to primary labels, and the localization analysis should be reliable. The branes with labels 0 and ∂X are at $x = 0$, the ones with label ϕ_i are at $x = \pi R$. The branes with labels φ^k are localized at points equally spread over the interval. All these branes have in addition a D1 component. The circle-inspired Fourier analysis cannot be trusted for the twisted sector branes and indeed gives contradictory results. The orientifold choices correspond to the four distinct choices of the Klein bottle current, $K = 0, \partial X, \phi_1$ and ϕ_2. For $K = 0$ we get K_{+++}, and we find two O0$_+$-planes at the orbifold points plus two O1$_+$-planes; For $K = \partial X$ (K_{++-}) we get two O0$_-$ planes at the orbifold points, and again two O1$_+$-planes. For $K = \phi_1$ or ϕ_2 (K_{-+0}) we get two coincident O0$_+$-planes at $x = \frac{1}{2}\pi R$, plus an (O1$_+$ + O1$_-$) configuration. For K_{+++} all CP-groups are SO. For K_{++-} the CP groups of boundaries $0, \partial X, \phi_i, \sigma_i$ and τ_i become unitary, while all others remain SO. For K_{-+0} all CP groups are unitary, except for $\varphi^k, k = N/2$ and the twist fields with either label 1 or 2, for which we find orthogonal groups. The group type for the φ^k branes is easy to understand: if the O-planes are in the middle of the orbifold line segment, they conjugate the branes mutually, except the brane in the center, which is self-conjugate. If the planes are on the endpoints, they conjugate all φ-branes to their orbifold image, *i.e.* to themselves, so that they are self-conjugate. A clear geometric picture suggests itself. Given a choice for the orbifold plane, there are two choices for the orientifold plane: on top of it, or orthogonal to it. The first choice leads to K_{+++} and K_{++-} and mostly self-conjugate branes, the second to K_{-+0} and mostly conjugate brane pairs. The proper geometric interpretation of the CP groups of the eight "special" branes (those not

labelled by φ^k) is somewhat less intuitive.

- N even, Twist automorphism: This is an exceptional invariant, and we obtained the boundary and crosscap coefficients numerically for small values of N, up to the sign ambiguity described above. Given the Klein bottle amplitude (K_{+-0}) one can easily compute the crosscap coefficients for all N: $C(0) = C(\partial\phi) = \frac{1}{2}$, $C(\varphi^{2k+1}) = \frac{1}{\sqrt{N}}, C(\varphi^{2k}) = 0$ This implies an O0$_+$ plane at $x = 0$ and and O0$_-$ plane at $x = \pi R$. In addition there are two O1$_+$ planes. All CP groups are orthogonal. We have no explicit formula for the reflection coefficient for arbitrary N, although it could be obtained in principle using the methods of [42] applied to the twist orbifold of the $c = 1$ orbifold (which is the $c = 1$ orbifold with four times the value of N). In the absence of such a formula it is difficult to discuss brane positions with these methods. There is also no canonical labelling of the boundary states.

- N odd, Charge conjugation invariant. The discussion of brane positions is identical to the one for even N, except that there is no brane in the middle. There are four possible choices for the Klein bottle current, but $K = \partial X$ and $K = \phi_2$ are known to be identical to $K = 0, K = \phi_1$ respectively, up to interchange of branes [11]. For $K = 0$ (K_{+-0}) the O-plane configuration is as for even N, and all CP groups are orthogonal except those of the ϕ_i, σ_i and τ_i branes, which are unitary. For $K = 1$ (K_{-+0}) the O-plane configuration is also the same as for even N, and all CP-groups are unitary except the ones labelled by σ_i, τ_i ($i = 1$ or 2), which are orthogonal. Apart from the usual eight special branes, these results are analogous to those for even N.

- N odd, Diagonal invariant. Here the formulas of [11] apply. The boundary states are labelled by orbits of the simple current ϕ_1. There are $N+1$ branes, two for each label φ^k with k even, one labelled by 0, and one labelled by a twist field. Boundary "0" is localized at the orbifold endpoints, the twist field boundary is not localizable, and all other boundaries occupy two symmetric positions on both sides of the center. There are two orientifold projections, distinguished by opposite crosscap signs. One of them yields K_{+++}, and all CP-groups are SO. The other yields K_{++-}, and all groups are SO except the one of the twist field boundary, which is symplectic. The O-plane configuration consists of two O1$_+$ planes, plus two O0$_+$ (for K_{+++}) and two O0$_-$ (for K_{++-}) planes located at the center. The fact that the boundaries are self-conjugate is understood as a consequence of the fact that each is symmetrically located on each

side of the O0-plane. Note however that the picture seems rather different than for even N, where the O0 planes are at the orbifold points. Note also that in this case the caveat regarding signs of the coefficient applies. If we modify all coefficients $C(k)$ by a factor $(-)^{k/2}$ the O-plane positions are as for even N (however, the brane positions, which also change, are still different than they are for even N).

Finally we can extract from [10] the crosscap and boundary coefficients for $R^2 = \alpha' p/q$, pq odd, but only up to signs, as explained above. For the crosscaps the Fourier transformations splits naturally into two sums, one proportional to $1/\sqrt{q}$ and the other to $1/\sqrt{p}$. The first gives O0 planes at multiples of $1/q$ of the full radius, the second at multiples of $1/p$, with signs depending on the case considered. These sums are completed by including $C(0) - C(\partial X)$, $C(\phi_1) + C(\phi_2)$ in one of the sums and $C(0) + C(\partial X)$, $C(\phi_1) - C(\phi_2)$ in the other, in agreement with the foregoing discussion. The result can be interpreted either in terms of a circle of radius $R^2 = \alpha' p/q$ or in terms of a circle of radius $R^2 = \alpha' q/p$. The first possibility corresponds a q-fold identification of the orbifold line, the second to a p-fold identification. In the first case the planes originating from the first Fourier sum are at the endpoints, whereas those from the second one are distributed equally on p points of the reduced line segment. It is natural to regard the latter as rational CFT realizations of D1 branes.

For $Z_C(p, q)$ two orientifold projections were found in [10], that differ by interchanging p and q. The O-plane charges are alternating for one of the Fourier sums, and identical for the other. On the reduced orbifold line segment this can be interpreted as a configuration with two $\mathrm{O0}_+$ planes at the end, plus one $\mathrm{O1}_+$ and one $\mathrm{O1}_-$ plane, and a configuration with one $\mathrm{O0}_+$ plane and one $\mathrm{O0}_-$ plane at the end, plus two $\mathrm{O1}_+$ planes (two, because odd and even points are to be identified with different O1-planes, as in the previous case). For $Z_D(p, q)$ there are also two orientifold projections, this time differing by signs in the crosscap coefficients, that flip the two Fourier sums with respect to each other. Using the same interpretation we now get two $\mathrm{O0}_+$ planes at the end, combined with either two $\mathrm{O1}_+$ or two $\mathrm{O1}_-$ planes. All this is identical to the results for odd, integer radius, except for the positions of the two $\mathrm{O0}_+$ planes. But precisely these positions are affected by the unknown signs. This particular kind of simple current MIPF (generated by a $\mathbf{Z}_4$-current with fixed points) does not appear in the circle theory and hence the correctness of these signs cannot be tested using brane localization on the circle.

3.2 Orientifolds for arbitrary radius

In this subsection we argue that the four Klein bottle amplitudes that we obtained in the previous subsection are the only possible ones for arbitrary radius. Thus, we find all possible orientifold maps, that is maps that project out states that are not invariant under the exchange $z \leftrightarrow \bar{z}$. Since the orientifold transformation of X_L and X_R must square to the identity, the oscillators transform like $a_n \to \bar{a}_n$. The only freedom left is in the operators coming from the z independent parts in the expansion of X_L and X_R.

The standard orientifold projection [6] corresponds to the map $X_L \to X_R$, giving rise to the amplitude (see (3.5))

$$K_{+++} = \frac{1}{2}\frac{1}{\eta(2i\tau_2)}\sum_{k\in\mathbf{Z}} q^{\frac{\alpha'}{2}\left(\frac{k}{R}\right)^2} + \frac{1}{2}\frac{1}{\eta(2i\tau_2)}\sum_{m\in\mathbf{Z}} q^{\frac{1}{2\alpha'}(mR)^2} + 2\sqrt{\frac{\eta}{\theta_4}}\ . \tag{3.7}$$

The first two terms arise from a trace over the states in the untwisted sectors. In the first of these two the orientifold map is inserted, and only the states with zero winding contribute. The second is the one with both the orientifold and orbifold map inserted, and since the KK momentum is not invariant under reflections, only states with no KK momentum contribute. The last term comes from the two twisted sectors; they both contribute in the same amount.

The first variation is to let the operators that create the ground states in the twisted sectors acquire a minus sign under the orientifold transformation. This will result in a minus sign in the last term of (3.5), giving

$$K_{++-} = \frac{1}{2}\frac{1}{\eta(2i\tau_2)}\sum_{k\in\mathbf{Z}} q^{\frac{\alpha'}{2}\left(\frac{k}{R}\right)^2} + \frac{1}{2}\frac{1}{\eta(2i\tau_2)}\sum_{m\in\mathbf{Z}} q^{\frac{1}{2\alpha'}(mR)^2} - 2\sqrt{\frac{\eta}{\theta_4}}\ . \tag{3.8}$$

In order to understand which other possible maps are allowed, one has to consider the way the various sectors interact. The untwisted sectors combine according to

$$(+,n)(+,m) = (+,n+m)\ . \tag{3.9}$$

Apart from the standard projection, this equation allows for the map

$$\begin{aligned} X_L &\to X_R + \frac{\pi}{2}\frac{\alpha'}{R} \\ X_R &\to X_L - \frac{\pi}{2}\frac{\alpha'}{R}\ , \end{aligned} \tag{3.10}$$

that changes the sign of the states in the sectors with odd winding. Because twisted sectors combine like

$$(-,\text{odd})(-,\text{even}) = (+,\text{odd}) \ , \tag{3.11}$$

consistency requires that the two twisted sectors contribute with opposite sign after the projection. The resulting amplitude is

$$K_{+-0} = \frac{1}{2}\frac{1}{\eta(2i\tau_2)}\sum_{k\in\mathbf{Z}} q^{\frac{\alpha'}{2}\left(\frac{k}{R}\right)^2} + \frac{1}{2}\frac{1}{\eta(2i\tau_2)}\sum_{m\in\mathbf{Z}} (-1)^m\, q^{\frac{1}{2\alpha'}(mR)^2} \ , \tag{3.12}$$

where the contribution from the twisted sectors cancels.

¿From T-duality one can then obtain the last consistent map,

$$\begin{aligned} X_L &\to X_R + \frac{\pi}{2}R \\ X_R &\to X_L + \frac{\pi}{2}R \ , \end{aligned} \tag{3.13}$$

that changes the sign of states with odd KK momentum. Since $X \to X + \pi R$, the twisted ground state localized in $X = 0$ is swapped for the one localized in $X = \pi R$. This means that 0 and πR are no longer fixed points of the orientifold map, whose eigenstates are now localized in $X = \pm\frac{\pi}{2}R$. Moreover, since the trace over the twisted states vanishes after the projection, there is no contribution from the twisted sectors to the Klein bottle amplitude, whose form is

$$K_{-+0} = \frac{1}{2}\frac{1}{\eta(2i\tau_2)}\sum_{k\in\mathbf{Z}} (-1)^k\, q^{\frac{\alpha'}{2}\left(\frac{k}{R}\right)^2} + \frac{1}{2}\frac{1}{\eta(2i\tau_2)}\sum_{m\in\mathbf{Z}} q^{\frac{1}{2\alpha'}(mR)^2} \ . \tag{3.14}$$

In order to understand why the choice $\epsilon_1 = \epsilon_2 = -1$ in eq. (3.5) is not allowed, we have to analyze the transverse channel. We only need to consider the untwisted sector, so we concentrate on the case K_{--0}. The Klein bottle in the direct channel depends on $2i\tau_2$, the modulus of the doubly-covering torus. The Klein bottle in the transverse channel is obtained performing an S modular transformation on the modulus of the doubly-covering torus, that is writing the amplitude in terms of $\ell = 1/2\tau_2$, the proper time in the transverse channel, describing a closed string propagating between two orientifold planes. The end-

result of this transformation (see for instance [7] for a review) is

$$\begin{aligned}
\tilde{K}_{++\pm} &= \frac{1}{2}\frac{1}{\eta(i\ell)}R\sqrt{\frac{2}{\alpha'}}\sum_{k\in\mathbf{Z}} q^{\frac{1}{\alpha'}(kR)^2} + \frac{1}{2}\frac{1}{\eta(i\ell)}\frac{1}{R}\sqrt{2\alpha'}\sum_{m\in\mathbf{Z}} q^{\alpha'(\frac{m}{R})^2} \pm 2\sqrt{\frac{\eta}{\theta_2}} \\
\tilde{K}_{+-0} &= \frac{1}{2}\frac{1}{\eta(i\ell)}R\sqrt{\frac{2}{\alpha'}}\sum_{k\in\mathbf{Z}} q^{\frac{1}{\alpha'}(kR)^2} + \frac{1}{2}\frac{1}{\eta(i\ell)}\frac{1}{R}\sqrt{2\alpha'}\sum_{m\in\mathbf{Z}} q^{\alpha'(\frac{m+1/2}{R})^2} \\
\tilde{K}_{-+0} &= \frac{1}{2}\frac{1}{\eta(i\ell)}R\sqrt{\frac{2}{\alpha'}}\sum_{k\in\mathbf{Z}} q^{\frac{1}{\alpha'}(k+1/2)^2R^2} + \frac{1}{2}\frac{1}{\eta(i\ell)}\frac{1}{R}\sqrt{2\alpha'}\sum_{m\in\mathbf{Z}} q^{\alpha'(\frac{m}{R})^2} \\
\tilde{K}_{--0} &= \frac{1}{2}\frac{1}{\eta(i\ell)}R\sqrt{\frac{2}{\alpha'}}\sum_{k\in\mathbf{Z}} q^{\frac{1}{\alpha'}(k+1/2)^2R^2} + \frac{1}{2}\frac{1}{\eta(i\ell)}\frac{1}{R}\sqrt{2\alpha'}\sum_{m\in\mathbf{Z}} q^{\alpha'(\frac{m+1/2}{R})^2} \ ,
\end{aligned} \tag{3.15}$$

where here $q = e^{-2\pi\ell}$. The states that contribute to the Klein bottle amplitude in the transverse channel are closed-string states propagating between two orientifold planes. In $\tilde{K}_{++\pm}$ only states with even KK momentum or even winding contribute in the untwisted sector. In $\tilde{K}_{+-0}$ only states with odd KK momentum or even winding contribute. This is consistent with the direct channel, since K_{+-0} projects out only states with odd winding. The same is valid for $\tilde{K}_{-+0}$, where only states with even KK momentum and odd winding contribute, while in the direct channel the states with odd KK momentum are projected out. Finally, in the case of $\tilde{K}_{--0}$ states with odd KK momentum and states with odd winding contribute to the transverse amplitude, but these states are both projected out by K_{--0}, so that this Klein bottle projection is not consistent.

¿From the transverse channel analysis one can also derive the position of the orientifold planes for the various Klein bottles. In all cases, namely the standard orientifold projection $X_L \to X_R$, and the ones given in (3.10) and (3.13), the map has no fixed points, and this corresponds to introducing O1-planes. Once the orbifold map $X \to -X$ is implemented, all these maps develop fixed points, where O0-planes are located. This means that all these amplitudes describe a configuration of two O1 planes, and two O0 planes located at the fixed points of the 'orientifold+orbifold' (Ωr) map. The twisted sector corresponds in the transverse channel to closed string states propagating between an O1 plane and an O0 plane.

In the case of $K_{++\pm}$, the Ω and Ωr maps are respectively $X \to X$ and $X \to -X$, and the two O1 planes have the same tension, as well as the two O0 planes. The two sign options for the twisted sector correspond to the fact that the tension of the O0 planes can be positive

or negative with respect to the tension of the O1 planes[5]. Since the Ωr map has fixed points in 0 and πR, the position of the two O0-planes coincides with the two fixed points of the orbifold. Applying the T-duality transformation $X = X_L + X_R \to X' = X_L - X_R$, and $R \to R' = \alpha'/R$, one can see that these two Klein bottles are both self-T-dual. In the case of K_{+-0}, the Ω projection is given in (3.10), and it maps X to itself, meaning that the two O1 planes have the same tension. The Ωr projection maps X to $-X$, so that the O0-planes are located at the fixed points of the orbifold. In order to determine the tension of the O0-planes, one has to consider how the Ωr transformation acts on the T-dual coordinate. In this case one has $X' \to X' + \pi R'$, and the shift in the dual coordinate is a manifestation of the fact that the two O0-planes have opposite tension. The Klein bottle amplitude corresponds thus to the configuration $\mathrm{O1}_+ \oplus \mathrm{O1}_+ \oplus \mathrm{O0}_+ \oplus \mathrm{O0}_-$, and the twisted sector cancels because of the opposite contribution from the two orbifold fixed points, where the O0-planes are located. Finally, in the case of K_{-+0}, the Ω projection is given in (3.13), and it maps X to $X + \pi R$, so that the two O1-planes have opposite tension, while Ωr maps X to $-X = \pi R$, so that the O0-planes are located in the middle of the orbifold segment. In this case the twisted sector cancels separately in any of the two (coincident) orientifold fixed points. T-duality maps this configuration to the previous one.

In all these cases the locations and charges of the O0 planes and the charges of the O1 planes agree with the results obtained from the CFT analysis in subsection 3.1.4, except those for the $K_{++\pm}$ Klein bottle of the D-invariants for odd N. In that case the charges are the same, but the two $\mathrm{O0}_+$ planes were found in the center rather than at the edges. But this was precisely a non-Cardy case, where the signs of the crosscap coefficients (crucial for the precise location) are not determined.

The foregoing discussion was for arbitrary radius, and seemed to rely in all cases on the standard orbifold map $X \to -X$. At rational radii the various orientifolds occur in combinations with specific partition functions, which require different orbifold maps, discussed in the appendix. This changes the map r in the foregoing discussion, and hence its fixed points. However, it also changes Ωr, whose fixed points determine the O-planes. It is easy to see that both modifications cancel, so that the relative position of orbifold fixed points and O0-plane positions remains unchanged. Note that nothing in the analysis in this

[5]The K_{++-} case is analogous to the six-dimensional brane supersymmetry breaking model of [43], in which the O5-planes and the O9-planes have opposite tension.

section imposed any relation between the orientifold map and the orbifold map in rational points. This relation was found in subsection 3.1.3 and makes use of OPE's involving distinct twist fields. We did not consider twist fields in this section, and furthermore for non-rational radius they cannot be distinguished, hence there is no reason to expect such a relation to emerge.

3.3 Orientifolds of exceptional MIPFs

As mentioned in section 3.1, the orbifold has exceptional MIPFs, constructed using an automorphism, ω, which leaves the fusion coefficients invariant: ${N_{ij}}^{k} = {N_{\omega(i)\omega(j)}}^{\omega(k)}$. The exceptional torus partition functions obtained from the chiral algebra of the CFT at square radius $R^2 = \alpha' pq$, with p and q odd prime numbers, are

$$T = \sum \chi_i \delta_{i\omega(j)} \bar{\chi}_j \ , \quad T = \sum \chi_i C_{i\omega(j)} \bar{\chi}_j \ . \tag{3.16}$$

In [10], these two invariants were called "diagonal + automorphism" (D+A) and "Cardy + automorphism" (C+A) respectively. Geometrically, these two tori describe a free boson compactified on an orbifold of radius $R^2 = \alpha' p/q$ and its T-dual.

We first review the results of [10] about orientifold projections. In the D+A case, the trivial Klein bottle, that is $K_i = 1$ for all the fields that couple diagonally on the torus, is allowed. A second Klein bottle is also allowed, with $K_i = -1$ for the twist fields and $K_i = 1$ for the other diagonal fields. In the C+A case, surprisingly the trivial choice $K_i = 1$ for all the fields coupling diagonally on the torus is not allowed. There are two Klein bottles. One has $K_{\phi_k} = -1$ when k is an odd multiple of p and $K_i = 1$ otherwise, and the other is obtained exchanging p with q.

Looking at the resulting amplitudes as functions of the orbifold radius, one realized that these Klein bottles are precisely the ones obtained in section 3.2, for a bosonic string compactified on a circle of square radius $R^2 = \alpha' p/q$. In particular, the Klein bottles of the D+A modular invariant are K_{+++} and K_{++-}, while the Klein bottles of the C+A modular invariant are K_{+-0} and K_{-+0}. Since the twisted sector is not diagonal for the C modular invariant, this is the only possibility that is allowed in light of the results of the previous subsection, and thus the results of [10] are completely consistent with the orientifold projections that are allowed for arbitrary radius.

4 Conclusions

We have identified four distinct orientifold projections for the $c = 1$ orbifolds. Geometrically, they can be described most easily starting from the O0 plane configurations of the T-dual circle. The $(\mathrm{O0}_+, \mathrm{O0}_-)$ configuration has only one axis of symmetry, namely the line through the O0-planes. Hence the orbifold and O-plane directions must line up, and the orbifold O0 planes are at its endpoints. The configuration $(\mathrm{O0}_+, \mathrm{O0}_+)$ has two axes of symmetry, and the orbifold reflection line is either on top of or orthogonal to the orientifold line. Then the O0-planes are respectively at the endpoints or on top of each other in the center.

In the circle theory on can distinguish two T-dual orientifold maps, one of the form $X_L \to +X_R + \mathrm{const}$ and one of the form $X_L \to -X_R + \mathrm{const}$. The former has fixed points in $X_L - X_R$, but not in $X_L + X_R$, whereas for the latter it is just the other way around. Therefore the former gives rise to O1 planes on the circle and the latter to O0-planes on the T-dual circle. The orbifold map (which has fixed points both in $X_L - X_R$ and $X_L + X_R$) transforms the two types of orientifold maps into each other, so that both O1 and O0 planes are present. Inspection of the transverse channel show that the charges of the O1 planes are identical if the orbifold fixed plane and the orientifold plane coincide, whereas they are opposite if these fixed planes are orthogonal. Allowing for an additional relative sign between the O1 and O0 planes then gives a total of four configurations (since the overall sign is irrelevant): $(\mathrm{O1}_+ \oplus \mathrm{O1}_+) \oplus (\mathrm{O0}_+ \oplus \mathrm{O0}_+)$, $(\mathrm{O1}_+ \oplus \mathrm{O1}_+) \oplus (\mathrm{O0}_- \oplus \mathrm{O0}_-)$, $(\mathrm{O1}_+ \oplus \mathrm{O1}_+) \oplus (\mathrm{O0}_+ \oplus \mathrm{O0}_-)$ and $(\mathrm{O1}_+ \oplus \mathrm{O1}_-) \oplus (\mathrm{O0}_+ \oplus \mathrm{O0}_+)$. This argument also shows why a fifth logical possibility, $(\mathrm{O1}_+ \oplus \mathrm{O1}_-) \oplus (\mathrm{O0}_+ \oplus \mathrm{O0}_-)$, cannot occur.

This intuitive argument was worked out in detail in section 3.2, and is backed up by the complete solution for U-NIMreps for rational CFT. The latter classification can be done exhaustively, but this is necessarily limited to a few rational points. We have shown that all four orientifolds are realized in all rational points, although in rather different ways. We have also shown how a known, but initially surprising solution at $R^2 = \alpha' p/q$ fits in perfectly with the continuum.

At arbitrary R we cannot rigorously rule out additional solutions, but in view of the agreement between the continuous R and the rational CFT descriptions, any deviations would be quite surprising.

A few open problems remain. While all methods agree on the O-plane charges, there

is a discrepancy on their precise positions in one case, interestingly precisely the case were the CFT results are least reliable. Secondly, the nature of the duality between diagonal and conjugation invariants of the rational orbifolds needs to be clarified. Finally, in the geometric description, applied to rational radii, the link between the choice among those two invariants and the orientifold map is not manifest.

Acknowledgments

We would like to thank Christoph Schweigert and Yassen Stanev for discussions and Herman Verlinde for clarifying some points in [2]. A.N.S. wishes to thank IMAFF-CSIC, Madrid, where part of this work was done, for hospitality. The work of F.R. and A.N.S. has been performed as part of the program FP 52 of the Foundation for Fundamental Research of Matter (FOM), and the work of T.P.T.D. and A.N.S. has been performed as part of the program FP 57 of FOM. The work of B.G.-R. and A.N.S. has been partially supported by funding of the Spanish "Ministerio de Ciencia y Tecnología", Project BFM2002-03610.

Appendix: Orbifold maps

Here we will discuss how the various partition functions enumerated in section 3.1.2 can be obtained from the circle theory. The standard description of orbifolds starts with a circle theory, from which the Z_2-symmetry $X \to -X$ is modded out. It is not hard to see that applying this map to the circle theories $Z(p,N)$ or $Z(N/p,N)$ (with $1 \leq p \leq \sqrt{N}$ one obtains in both cases the orbifold theory $Z_{...}(p,N)$. The problem is that in the rational case the orbifold partition function has an additional label D, C or ij. Since the distinction is not made by the T-duality of the circle, there must be more than one way to do the orbifold map. Obviously one can generalize it to $X \to a - X$, *i.e.* rotating the plane of reflection, but this does not have the desired effect.

It turns out that one must consider the chiral orbifold map

$$X_L \to a_L - X_L \ ; \quad X_R \to a_R - X_R \ .$$

On the vertex operators $V(k)$ corresponding to the fields φ^k the only effect is a phase between the two terms of which they consist; but the effect is more important for the generators of the chiral algebra and the fields ϕ_1 and ϕ_2, which make the difference between the rational and the non-rational case (we will ignore the twist fields here, since the difference between the various partition functions is already clear in the untwisted sector). Note that the circle theory operators from which ϕ_1 and ϕ_2 originate, which have $k = \pm N$ and chiral ground state multiplicity 2, appear in an identical way for a circle and its T-dual.

The dependence of these vertex operators on a_L and a_R is as follows for the chiral algebra generators

$$W_L = V(R,0) + e^{ia_L \frac{2R}{\alpha'}} V(-R,0)$$

and

$$W_R = V(0,R) + e^{ia_R \frac{2R}{\alpha'}} V(0,-R) \ ,$$

where

$$V(r,s) = e^{i\frac{2r}{\alpha'}X_L} e^{i\frac{2s}{\alpha'}X_R} \ .$$

For the other four circle operators with $\mid r \mid = \mid s \mid = \frac{1}{2}R$ we get two invariant combinations

$$V^A = V(+,+) + e^{i\frac{R}{\alpha'}(a_L + a_R)} V(-,-)$$

and

$$V^B = V(+,-) + e^{i\frac{R}{\alpha'}(a_L - a_R)} V(-,+) \ ,$$

with the arguments "+" and "−" denoting $+R/2$ and $-R/2$ respectively.

The operators V^A and V^B are Virasoro-degenerate, but are distinguished by the chiral algebra operators W_L and W_R. In order to relate these operators to a partition function interpretation we need to combine V^A and V^B into chiral algebra eigenstates. For this we need the OPE of these operators, and here an important rôle is played by the cocycle factors that should be added to these operators [44]. In this case these cocycles can be conveniently represented by Pauli matrices $(\sigma_3)^m(\sigma_1)^n$ where m and n are the winding and momentum quantum numbers of the operator. It is easy to see that W_L and W_R acquire a factor $\sigma_3(\sigma_1)^N$, $V(+,+)$ and $V(-,-)$ a factor $(\sigma_1)^N$ and $V(+,-)$ and $V(-,+)$ a factor σ_3. Hence for even N the cocycles do not change anything in comparison with the "naive" OPE. For arbitrary N the chiral algebra eigenstates are found to be

$$V(+,+) + e^{i\frac{R}{\alpha'}(a_L + a_R)} V(-,-) \pm (i^N e^{i\frac{R}{\alpha'}a_R} V(+,-) + i^N e^{i\frac{R}{\alpha'}a_L} V(-,+)) \ .$$

For even N this can be factorized as

$$\left[V_L(+) \pm e^{i\frac{R}{\alpha'}a_L} V_L(-)\right] \left[V_R(+) \pm e^{i\frac{R}{\alpha'}a_R} V_R(-)\right] \ ,$$

with correlated signs in the two factors; for odd N it cannot be factorized. For these operators to have sensible reality properties, a_L and a_R must be quantized as multiples of $\alpha'\pi/R$, the allowed positions in the rational CFT description (these are precisely the allowed brane positions on the circle; any other value would not allow a rational CFT interpretation). Then one finds that for N even the operators are real, and for N odd they are each others conjugate, in agreement with the modular matrix S [2].

For the standard case $a_L = a_R = 0$, and for N even, the operators have the expected "cos cos" and "sin sin" form that is indicative of the diagonal invariant. By choosing $a_L = 0$, $a_R = \alpha'\pi/R$ one can change this to a "cos sin" and "sin cos" form, corresponding to the conjugation invariant. For odd N the results are similar. The operators for $a_L = a_R = 0$ can be written in the form "cos cos $\pm i$ sin sin" and they change to "cos sin $\pm i$ sin cos" for $a_L = 0, a_R = \alpha'\pi/R$. These two cases should correspond, respectively, to the diagonal and charge conjugation invariant of the odd N orbifold. To get the heterotic orbifold invariants we may choose $a_L = 0, a_R = \alpha'\pi/2R$. Note that this value for a_R does not belong to the set of allowed positions, but it is an allowed position for the orbifold with twice the value

of R. The heterotic theory is obtained as a chiral algebra extension of the latter CFT. The term in the partition function corresponding to V^A and V^B has multiplicity 2, and it is a simple current fixed point, which cannot be resolved using the orbifold data alone. Hence the reality properties of these operators are not determined.

The cocycle factors are also needed in the operators Ω that implement the various orientifold maps on the vertex operators. In some cases one has to include a factor σ_3 in these operators, which affects the result only for odd N and only when acting the operators V^A and V^B.

References

[1] R. Dijkgraaf, E. Verlinde and H. Verlinde, Commun. Math. Phys. **115**, 649 (1988).

[2] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, Commun. Math. Phys. **123**, 485 (1989).

[3] P. Ginsparg, Nucl. Phys. B **295**, 153 (1988).

[4] E. B. Kiritsis, Phys. Lett. B **217**, 427 (1989).

[5] I. Runkel and G. M. Watts, Fortsch. Phys. **50**, 959 (2002) [arXiv:hep-th/0201231].

[6] G. Pradisi and A. Sagnotti, Phys. Lett. B **216** (1989) 59.

[7] C. Angelantonj and A. Sagnotti, Phys. Rept. **371**, 1 (2002) [Erratum-ibid. **376**, 339 (2003)] [arXiv:hep-th/0204089].

[8] A. Cappelli and G. D'Appollonio, JHEP **0202** (2002) 039 [arXiv:hep-th/0201173].

[9] A. N. Schellekens, Int. J. Mod. Phys. A **14**, 1283 (1999) [arXiv:math.qa/9806162].

[10] A. N. Schellekens and N. Sousa, Int. J. Mod. Phys. A **16**, 3659 (2001) [arXiv:hep-th/0009100].

[11] J. Fuchs, L. R. Huiszoon, A. N. Schellekens, C. Schweigert and J. Walcher, *Boundaries, crosscaps and simple currents*, Phys. Lett. B **495**, 427 (2000) [arXiv:hep-th/0007174].

[12] B. Gato-Rivera and A. N. Schellekens, *Complete classification of simple current modular invariants for (Z(p))**k*, Commun. Math. Phys. **145**, 85 (1992); M. Kreuzer and A.N. Schellekens, *Simple currents versus orbifolds with discrete torsion: A Complete classification*, Nucl. Phys B411, 97 (1994) [arXiv:hep-th/9306145].

[13] L. R. Huiszoon, *D-branes and O-planes in string theory: An algebraic approach*, PhD thesis.

[14] J. Fuchs, I. Runkel and C. Schweigert, *TFT construction of RCFT correlators. III: Simple currents* [arXiv:hep-th/0403157].

[15] A. N. Schellekens and Y. S. Stanev, JHEP **0112**, 012 (2001) [arXiv:hep-th/0108035].

[16] N. Sousa and A. N. Schellekens, Nucl. Phys. B **653**, 339 (2003) [arXiv:hep-th/0210014].

[17] P. Di Francesco and J.-B Zuber, Nucl. Phys. **B338** (1990) 602.

[18] G. Pradisi, A. Sagnotti and Ya.S. Stanev, Phys Lett B354 (1995) 279; Phys. Lett. B381 (1996) 97 [arXiv:hep-th/9603097].

[19] R. Behrend, P. Pearce, V. Petkova and J-B. Zuber, Phys. Lett. B **444**, 163 (1998) [arXiv:hep-th/9809097]; Nucl. Phys. B **579**, 707 (2000) [arXiv:hep-th/9908036].

[20] M. Bianchi, G. Pradisi and A. Sagnotti, Nucl. Phys. B **376** (1992) 365.

[21] A. Dabholkar and J. Park, Nucl. Phys. B **477** (1996) 701 [arXiv:hep-th/9604178].

[22] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y. S. Stanev, Phys. Lett. B **387** (1996) 743 [arXiv:hep-th/9607229].

[23] E. Witten, JHEP **9802** (1998) 006 [arXiv:hep-th/9712028].

[24] J. Fuchs and C. Schweigert, Nucl. Phys. B **530**, 99 (1998) [arXiv:hep-th/9712257].

[25] O. Bergman, E. G. Gimon and S. Sugimoto, JHEP **0105** (2001) 047 [arXiv:hep-th/0103183].

[26] M. Davidse, *Compactifications, boundary states and tadpole equations*, Masters thesis (unpublished).

[27] L. R. Huiszoon, A. N. Schellekens and N. Sousa, Nucl. Phys. B **575**, 401 (2000) [arXiv:hep-th/9911229].

[28] J. Fuchs and C. Schweigert, Phys. Lett. B **414**, 251 (1997) [arXiv:hep-th/9708141]; Nucl. Phys. B **558**, 419 (1999) [arXiv:hep-th/9902132]; Nucl. Phys. B **568**, 543 (2000) [arXiv:hep-th/9908025]; L. R. Huiszoon, A. N. Schellekens and N. Sousa, Phys. Lett. B **470**, 95 (1999) [arXiv:hep-th/9909114]; Nucl. Phys. B **575**, 401 (2000) [arXiv:hep-th/9911229]; L. R. Huiszoon and A. N. Schellekens, Nucl. Phys. B **584**, 705 (2000) [arXiv:hep-th/0004100].

[29] J. L. Cardy, *Boundary Conditions, Fusion Rules And The Verlinde Formula*, Nucl. Phys. B **324** (1989) 581.

[30] G. Pradisi, A. Sagnotti and Y. S. Stanev, Phys. Lett. B **354**, 279 (1995) [arXiv:hep-th/9503207].

[31] P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, Nucl. Phys. B **507**, 259 (1997) [arXiv:hep-th/9707068].

[32] G. Felder, J. Frohlich, J. Fuchs and C. Schweigert, J. Geom. Phys. **34** (2000) 162 [hep-th/9909030].

[33] L. R. Huiszoon, K. Schalm and A. N. Schellekens, Nucl. Phys. B **624**, 219 (2002) [arXiv:hep-th/0110267].

[34] I. Brunner, *On Orientifolds of WZW Models and their Relation to Geometry*, hep-th/0110219.

[35] C. Bachas, N. Couchoud and P. Windey, JHEP **0112**, 003 (2001) [arXiv:hep-th/0111002].

[36] A. Recknagel and V. Schomerus, Nucl. Phys. B **545**, 233 (1999) [arXiv:hep-th/9811237].

[37] G. Pradisi, A. Sagnotti and Y. S. Stanev, Phys. Lett. B **381**, 97 (1996) [arXiv:hep-th/9603097].

[38] S. Hamidi, C. Vafa Nucl. Phys. B **279**, 465 (1987).

[39] J. Fuchs, A. N. Schellekens and C. Schweigert, Nucl. Phys. B **437**, 667 (1995) [arXiv:hep-th/9410010].

[40] T. Gannon, P. Ruelle and M. A. Walton, arXiv:hep-th/9511089.

[41] G. Felder, J. Frohlich, J. Fuchs and C. Schweigert, Phys. Rev. Lett. **84**, 1659 (2000) [arXiv:hep-th/9909140].

[42] L. Birke, J. Fuchs and C. Schweigert, Adv. Theor. Math. Phys. **3**, 671 (1999) [arXiv:hep-th/9905038].

[43] I. Antoniadis, E. Dudas and A. Sagnotti, Phys. Lett. B **464** (1999) 38 [arXiv:hep-th/9908023].

[44] K. Itoh, M. Kato, H. Kunitomo and M. Sakamoto, Nucl. Phys. B **306**, 362 (1988).